Hello everyone, my topic today is the valuation of storage. This is a joint work with Kumar Muthuraman and Stathis Tompaidis.

Here are some examples of storage. Silos (sailou) for agricultural (/ˌægrɪ'kʌltʃərəl/ ) commodities, Tanks for oil, Caverns (/'kævən/) for natural gas and Lake reservoirs (/'rɛzɚ,vɔrs/) or dams for water.

\*\*\*\*\*I would like to use natural gas as an example to show what is important to the value of storage. Here is the plot of production and consumption of Natural gas from Jan 2001 to Aug 2014. If we take consumption as demand and production as supply, the demand is much more volatile than the supply. Demand peaks in winters because natural gas is used for heating. Intuitively, the price should be high in winters and low in summers.

This intuition is actually right. Here is the plot of future prices of natural gas. You can see a clear seasonality in the price. This gives an incentive to store natural gas in the summer and sell it in the winter. Assume the storage has no transaction cost, and then everyone will do the same thing. This will eliminate the mismatch between supply and demand. The price will be flat. There will be no value of storage at all. Therefore the transaction cost is essential to the value of storage. On the other hand, transaction cost shouldn’t be constant. Still take natural gas as an example. The more gas in the cavern, the higher pressure is needed to inject. The pump should use more energy which means the higher transaction cost. Therefore, the transaction cost should be a function of storage level. If the pump uses the purchased gas, it should also be a function of the price. This is the motivation why we consider the valuation of storage with non-trivial transaction cost.\*\*\*\*

\*\*\*There are two decisions to make at each time. At the spot price right now, do I buy, sell or hold? If I decide to sell, how much should I sell? If I decide to buy, how much should I buy? \*\*\*The valuation problem turns out to be a continuous time singular control problem. It can be further transferred into solving an HJB equation. HJB equation is a free boundary problem, which is very hard to solve. Moving boundary method is used tin 1 dimension. The idea is as following. Start with an initial guess and iteratively improves it until convergence. In other words, free boundary problem is transformed into a sequence of fixed boundary problems, which are easier to solve. \*\*\*There are 3 key points here. How to come up with a suitable initial guess. How to improve the boundary? How to solve fixed boundary problem? \*\*\*

\*\*\*\* However, moving boundary method can’t be used directly here because it is a 2 dimensional problem. In 1 dimension, only several points need to be found. However, in 2 dimensions, there are all kinds of shapes. Even we can slice 2 dimensions into lots of 1 dimension, we need to decide which direction should be chosen. Is it horizontal or vertical? How about diagonal? Therefore, from 1 to 2 dimensions is not trivial. Actually, generalizing moving boundary problem from 1 dimension to 2 is a major contribution in this paper. \*\*\*

Speaking of contributions let me give you an overview of the results. The first result of this paper is that fixed boundary problem is solved efficiently. This is very important for implementing moving boundary method. As I mentioned before, moving boundary method turns solving a free boundary problem into solving a sequence of fixed boundary problems. As long as solving fixed boundary problem can be handled easily, moving boundary method is worth doing. The second is what I just mentioned from 1 dimensional to 2. This consists of four things. \*\*\*\*\*A reasonable way of generating initial guess is proved. Which direction and how long should be moved is decided. Finally, the convergence is proved. Then we solve the valuation of storage using this generalized moving boundary method.\*\*\*\*

Here are the references. They are about commodity price, valuation of storage and moving boundary method.

One factor model is used here as the price dynamic. By Ito’s formula, log price is an OU process. Capital Q\_t is the storage level at time t. Capital L\_t is the cumulative injections and capital U\_t is cumulative withdrawals. Therefore, the change in storage dQ\_t is equal to the amount that injected minus the amount withdrawn. A policy is admissible as long as if Q\_t remains inside a given interval. Transaction costs are functions of both the log price and the amount in storage as I mentioned before.

Objective is to maximize discounted infinite-horizon cash flows. The first part is the money earned from selling and the second part is the cost of buying.

Use dynamic programming principal and Ito’s formula. The optimal value function satisfies this HJB equation. On the other hand, verification theorem shows that the solution of HJB equation is actually optimal.

I would like to talk more about HJB equation because the intuition of moving boundary method depends heavily on it. Assume the value function is known and the change of policy at one point x null q null won’t affect it. Now, epsilon is bought at price exponential x null. The average buying profit should be this expression. The first part is the discounted value earned in the future while the second part is the cost I need to pay right now. When epsilon goes to 0, it becomes the second term of HJB equation shown at the bottom of this slide. That is the reason why we call them holding profit, selling profit and buying profit. HJB equation at x null q null simply says that you can’t improve any more. When you can’t improve at all points, you are at optimal. That is all about HJB equation.

So the state space is divided into three regions. For holding region: holding is the best. For selling region, selling is the best. For buying region, buying is the best.

If those regions are given, namely the boundary being fixed, it is a fixed boundary problem. In the holding region, if we do a transformation like this, the equation will turns into Kummer equation. We can use hypergeometric functions to solve it like this. The coefficients A(q) and B(q) can be solved using boundary conditions on the buying and selling boundary.

Next, I would like to address the problem of initial guess. Two intuitions, when the price is high, sell and when the price is low, buy. Actually, a theorem is proved that when the price is high enough, selling is the optimal strategy. When there is no transaction cost at all, it is obviously true. In case that the transaction cost is non-zero, because transaction cost is bounded, when the price is high, the cost is negligible /'nɛɡlɪdʒəbl/. It should share the same result as the case without transaction cost. However, the second one is not right. Let me give a simple counterexample. Assume injecting cost is close to infinity. You will never buy no matter how low the price is. The reason for this is that the buying cost consists of price and transaction cost. When transaction cost is dominating, the second intuition is wrong.

Another structure is that under optimal policy, there doesn’t exist (x,q1) and (x,q2) that one buys at first point while sells at the second. The case in the figure won’t happen. It is obvious without transaction. When it is a downward trend, one sells. When it is an upward trend, one buys. The predicted trend tells them apart. (I don’t know how to say it) With transaction costs, one will not only consider the trend but also whether this price can cover my cost or not. Keep in mind that transaction costs won’t change the trend. Therefore, the predicted trend can still tells them apart.

Right now, we come into the moving boundary method. There are two challenges. 1. Because the second intuition fails, there is no easy initial guess for buying boundary. 2.As I said several times, it is a 2 dimensional problem. We need to know the direction and distance of the movement in order to turn this one into a 1 dimensional problem. Our solutions are inspired by the theorem I just showed you. When the price is high enough, the optimal thing is always selling. This means that the initial guess of selling boundary does exist like this. The green indicts selling and blue line is the mean reverting level. Therefore, for every storage level q, there has already been an initial guess for selling. In this case, there is no reason to not choose move along the price x with fixed q. I will talk more about how long should it move later. Once we decide which direction to move, we can generate the initial guess for buying boundary inside the moving boundary method. \*\*\*\*The idea is straightforward. Those points with positive buying profit will be taken as the initial guess for buying boundary. \*\*\*\*\*

Start with selling at a very high price. Move selling boundary down along x. We found there are points that have positive selling profit. Pick those points as the initial guess for selling boundary. Move buying boundary to them shown in red. (Buying boundary used to be none.) Move down selling boundary again. Move buying boundary. Move down selling. Move up and down buying. Do this until convergence.

Notice there is no coming back in moving boundary method. Overshooting must be avoided. How long should it move is addressed here. For selling boundary, consider the black line in the picture. The selling profit along this line is like this. Move to the maximum selling profit point, which is in green. It becomes this. The reason that the maximum is chosen is that in this way we can prove that boundary can be kept moving. It won’t stop. For buying boundary, the buying profit along this line is like this. Move both up and down to the local maximum buying profit point, which are in red. It becomes this.

\*\*\*\*\*In the proof of convergence, we show that each movement improves value function. Let me explain the idea using the case there is no selling boundary. Because it is very hard to see the difference between two consecutive steps, I use the initial guess and the 9th step, but the idea is the same. Those two points are at the same x level and they are on the selling boundary of second case. In this case, we directly sell it. Therefore the difference is the money gaining by selling the difference of two storage levels minus the transaction cost. However, in the first case, because the selling price is high above mean reverting level, it is a long time wait for selling. Therefore the difference between those two is small. This is actually true for all points on the same x level. On the other hand, for q = 0, because we will not buy anything, the value of all the points on the y axis is 0. Combining those two facts, the value of all points should be larger than the first case. \*\*\*\*\*

2. The boundaries can be kept moving. Details are omitted here.

Thank you!