Hello everyone, my topic today is the valuation of storage. This is a joint work with Kumar Muthuraman and Stathis Tompaidis.

Here are some examples of storage. Silos (sailou) for agricultural (/ˌægrɪ'kʌltʃərəl/ ) commodities, Tanks for oil, Caverns (/'kævən/) for natural gas and Lake reservoirs (/'rɛzɚ,vɔrs/) or dams for water.

I would like to use natural gas as an example to show what is important to the value of storage. Here is the plot of production and consumption of Natural Gas from Jan 2001 to Aug 2014. If we take consumption as demand and production as supply, the demand is much more volatile than the supply. Demand peaks in winters because natural gas is used for heating. Intuitively, the price should be high in winters and low in summers.

This intuition is right. Here is the plot of future prices of natural gas. You can see a clear seasonality in the price. This gives the incentive to store natural gas in the summer and sell it in the winter. Assume the storage has no transaction cost, and then everyone will do the same thing. This will eliminate the mismatch between supply and demand. The price will be flat. There will be no value of storage at all. Therefore, the transaction cost is essential to the value of storage. On the other hand, transaction cost should not be constant. Still take natural gas as an example. The more gas in the cavern, the higher pressure is needed to inject. The pump should use more energy which means the higher transaction cost. Therefore, the transaction cost should be a function of storage level. If the pump uses the purchased gas, it should also be a function of the price. This is the motivation why we consider the valuation of storage with non-trivial transaction cost.

There are two decisions to make at each time. At the spot price right now, do I buy, sell or hold? If I decide to sell, how much should I sell? If I decide to buy, how much should I buy? The valuation problem turns out to be a continuous time singular control problem. It can be further transferred into solving an HJB equation. HJB equation is a free boundary problem, which is very hard to solve. Moving boundary method is used to solve HJB equation numerically in 1 dimension. We want to generalize it to 2 dimensions to fit our problem because here we have two dimensions, price, and storage level. It is not trivial. In 1 dimension, only several points need to be found. However, in 2 dimensions, there are all kinds of shapes. Even we can slice two dimensions into lots of 1 dimension, we need to decide which direction should be chosen. Is it horizontal or vertical? How about diagonal? A major contribution of this paper is developing the 2-dimensional moving boundary method.

Here are the references. They are about commodity price, valuation of storage and moving boundary method.

One factor model is used here as the price dynamic. By Ito’s formula, log price is an OU process. Capital Q\_t is the storage level at time t. Capital L\_t is the cumulative injections and capital U\_t is cumulative withdrawals. Therefore, the change in storage dQ\_t is equal to the amount that injected minus the amount withdrawn. A policy is admissible as long as if Q\_t remains inside a given interval. Transaction costs are functions of both the log price and the amount in storage as I mentioned before.

Objective is to maximize discounted infinite-horizon cash flows. The first part is the money earned from selling and the second part is the cost of buying.

Use dynamic programming principal and Ito’s formula. The optimal value function satisfies this HJB equation. On the other hand, verification theorem shows that the solution of HJB equation is actually optimal.

I would like to talk more about HJB equation because the intuition of moving boundary method is highly dependent on it. Assume the value function is known and the change of policy at one point x null q null won’t affect it. Now, epsilon is bought at price exponential x null. The average buying profit should be this expression. The first part is the discounted value earned in the future while the second part is the cost I need to pay right now. When epsilon goes to 0, it becomes the second term of HJB equation shown at the bottom of this slide. That is the reason why we call them holding profit, selling profit and buying profit. HJB equation at x null q null simply says that you can’t improve any more. When you can’t improve at all points, you are at optimal. That is all about HJB equation.

So the state space is divided into three regions. For holding region: holding is the best. For selling region, selling is the best. For buying region, buying is the best.

We need to solve those three regions together with solving V. This is the reason why HJB equation is called a free boundary problem. If those regions are given, namely the boundary being fixed, it is a fixed boundary problem. Clearly fixed boundary problem is much easier than the free boundary one.

Next we would like to use the moving boundary method to solve this HJB equation numerically. The idea of moving boundary method is as following. Start with an initial guess and iteratively improves it until convergence. In other words, free boundary problem is transformed into a sequence of fixed boundary problems, which are easier to solve. There are 3 key points here. How to come up with a suitable initial guess? Can we solve fixed boundary problem efficiently? How to improve the boundary?

First, I would like to address the problem of initial guess. Two intuitions, when the price is high, sell and when the price is low, buy. When there is no transaction cost at all, it is obviously true. In case that the transaction cost is non-zero, because transaction cost is bounded, when the price is high, the cost is negligible /'nɛɡlɪdʒəbl/. It should share the same result as the case without transaction cost. Actually, a theorem is proved that when the price is high enough, selling is the optimal strategy. However, the second one is not right. Let me give a simple counterexample. Assume injecting cost is close to infinity. You will never buy no matter how low the price is. The reason for this is that the buying cost consists of price and transaction cost. When transaction cost is dominating, the second intuition is wrong. Our initial guess will be selling at a high price and doing nothing else where.

Next, we need to find an efficient way to solve the fixed boundary problem. This is very important for implementing moving boundary method. As I mentioned before, moving boundary method turns solving a free boundary problem into solving a sequence of fixed boundary problems. As long as solving fixed boundary problem can be handled easily, moving boundary method is worth doing. In the holding region, if we do a transformation like this, the equation will turn into Kummer’s equation. We can use hypergeometric functions to solve it like this. The coefficients A(q) and B(q) are the only unknowns here, and they can be solved using boundary conditions on the buying and selling boundary. In another word, it is as efficient as solving a one-dimensional problem.

At last, how should the boundary move? We need to know the direction and distance of the movement to turn this one into a one-dimensional problem. Our solutions are inspired by the initial guess. The initial guess is like this. The green indicts selling, and the blue line is the mean reverting level. Therefore, for every storage level q, there has already been an initial guess for selling. In this case, there is no reason not to choose move along the price x with fixed q. Once we decide which direction to move, we can generate the initial guess for buying boundary inside the moving boundary method. The idea is straightforward. Those points with positive buying profit will be taken as the initial guess for buying boundary. Then we can move both boundaries at the same time.

Next, I would like to talk about how long should it move. Because there is no coming back in the moving boundary method, overshooting must be avoided. For selling boundary, consider the black line in the picture. The selling profit along this line is like this. Move to the maximum selling profit point, which is in green. It becomes this. The reason that the maximum is chosen is that in this way we can prove that boundary can be kept moving. It will not stop. For buying boundary, the buying profit along this line is like this. Move both up and down to the local maximum buying profit point, which are in red. It becomes this.

In the proof of convergence, we show that each movement improves value function. PDE of \Delta V\_{n+1} and a figure.

2. The boundaries can be kept moving. Give some intuition. Take the selling boundary as an example. The boundary can move is equivalent to the value function decreases slower than the selling price – transaction cost. Because the PDE is harmonic, it can only achieve the Unfinished.

At this maximum point, the

Thank you!

I will talk more about how long should it move later.

Start with selling at a very high price. Move selling boundary down along x. We found there are points that have positive selling profit. Pick those points as the initial guess for selling boundary. Move buying boundary to them shown in red. (Buying boundary used to be none.) Move down selling boundary again. Move buying boundary. Move down selling. Move up and down buying. Do this until convergence.

In the proof of convergence, we show that each movement improves value function. Let me explain the idea using the case there is no selling boundary. Because it is very hard to see the difference between two consecutive steps, I use the initial guess and the 9th step, but the idea is the same. Those two points are at the same x level and they are on the selling boundary of second case. In this case, we directly sell it. Therefore, the difference is the money gaining by selling the difference of two storage levels minus the transaction cost. However, in the first case, because the selling price is high above mean reverting level, it is a long time wait for selling. Therefore, the difference between those two is small. This is true for all points on the same x level. On the other hand, for q = 0, because we will not buy anything, the value of all the points on the y-axis is 0. Combining those two facts, the value of all points should be larger than the first case.